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VIBRATION DAMPING BY A CONTINUOUS DISTRIBUTION OF UNDAMPED OSCILLATORS

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In this paper it is shown that a continuous spatial distribution of undamped secondary oscillators can produce damping in an otherwise undamped primary spring–mass system. Damping here refers to exact exponential decay, valid for all time, in the vibration of the primary system. The continuous distribution of secondary oscillators is the limiting case of a large number of discrete secondary oscillators, and thus this paper may be related to recent papers by Pierce *et al*. [1], Strasberg and Feit [2] and Weaver [3]. Wave propagation in systems with continuous distributions of attached oscillators has been considered by Zozulya and Rybak [4, 5].

The system considered here is shown in Figure 1. A primary mechanical oscillator with mass M and stiffness k_0 is connected to a set of secondary oscillators with mass $m(\xi)$ and stiffness $k(\xi)$. It is assumed that the secondary oscillators are *continuously distributed*; the variable ξ may be thought of as a dimensionless spatial co-ordinate which has values in the interval $0 < \xi < 1$. The distributed set of secondary oscillators is analogous in some ways to the distributed elastic foundation which is used in many structural applications.

The equations of motion for the system in Figure 1 are

$$M\ddot{x}_M(t) + k_0 x_M(t) + \int_0^1 \mathrm{d}\xi\, k(\xi)(x_M(t) - x(\xi, t)) = 0, \tag{1}$$

$$m(\xi)\ddot{x}(\xi, t) + k(\xi)(x(\xi, t) - x_M(t)) = 0, \qquad 0 < \xi < 1. \tag{2}$$

It is assumed that the initial conditions are

$$x_M(0) = x(\xi, 0) = \dot{x}(\xi, 0) = 0, \qquad \dot{x}_M(0) = v_0, \tag{3, 4}$$

and it is desired to derive the dynamic response of the system for $t > 0$.

Introducing the Laplace transform according to the usual definitions

$$\tilde{x}_M(s) = \int_0^\infty \mathrm{d}t\, x_M(t)\, \mathrm{e}^{-st}, \qquad \tilde{x}(\xi, s) = \int_0^\infty \mathrm{d}t\, x(\xi, t)\, \mathrm{e}^{-st} \tag{5, 6}$$

and applying the initial conditions given by equations (3) and (4), equations (1) and (2) become

$$M(s^2\tilde{x}_M(s) - v_0) + k_0\tilde{x}_M(s) + \int_0^1 \mathrm{d}\xi\, k(\xi)(\tilde{x}_M(s) - \tilde{x}(\xi, s)) = 0, \tag{7}$$

$$m(\xi)s^2\tilde{x}(\xi, s) + k(\xi)(\tilde{x}(\xi, s) - \tilde{x}_M(s)) = 0. \tag{8}$$

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From equations (7) and (8) there follows

$$\tilde{x}_M(s) = v_0 \Bigg/ \left[s^2 + \Omega_0^2 + s^2 \int_0^1 \mathrm{d}\xi \, \frac{\Omega^2(\xi)(m(\xi)/M)}{s^2 + \Omega^2(\xi)} \right] \tag{9}$$

and

$$\tilde{x}(\xi, s) = \frac{\Omega^2(\xi) v_0}{(s^2 + \Omega^2(\xi)) \left(s^2 + \Omega_0^2 + s^2 \int_0^1 \mathrm{d}\xi \, \frac{\Omega^2(\xi)(m(\xi)/M)}{s^2 + \Omega^2(\xi)} \right)}, \tag{10}$$

where

$$\Omega^2(\xi) = k(\xi)/m(\xi) \qquad \text{and} \qquad \Omega_0^2 = k_0/M. \tag{11, 12}$$

In order to obtain relatively simple inverse Laplace transforms of equations (9) and (10), it is assumed that the mass and stiffness of the distributed oscillators are given by the particular functions

$$m(\xi) = \mu M / [(1 - \xi)^2 + \xi^2] \tag{13}$$

and

$$k(\xi) = \mu k_0 \xi^2 / (1 - \xi)^2 [(1 - \xi)^2 + \xi^2], \tag{14}$$

where μ is a dimensionless constant. The mass $m(\xi)$ is finite everywhere, and the total distributed mass is

$$\int_0^1 \mathrm{d}\xi \, m(\xi) = \int_0^1 \mathrm{d}\xi \, \frac{\mu M}{(1 - \xi)^2 + \xi^2} \tag{15}$$

$$= \mu M \int_0^1 \frac{\mathrm{d}\xi}{(1 - \xi)^2} \frac{1}{1 + \xi^2/(1 - \xi)^2}. \tag{16}$$

Changing the variable of integration from ξ to $\theta = \xi/(1 - \xi)$, equation (16) becomes

$$\int_0^1 \mathrm{d}\xi \, m(\xi) = \mu M \int_0^\infty \mathrm{d}\theta \, \frac{1}{1 + \theta^2} \tag{17}$$

$$= \mu \pi M / 2, \tag{18}$$

so that the ratio of the total distributed mass to the mass M of the primary oscillator is $\mu\pi/2$. The stiffness $k(\xi)$ ranges from zero (no coupling to the primary oscillator) at $\xi = 0$ to infinity (rigid coupling to the primary oscillator) as $\xi \to 1$. The functions $m(\xi)$ and $k(\xi)$ in equations (13) and (14) are special choices, but they are physically plausible and they allow the interesting dynamic behavior of the system in Figure 1 to be obtained explicitly.

Using equations (13), (14) and (11), the integral term in the denominator of the right sides of equations (9) and (10) becomes

$$s^2 \int_0^1 \mathrm{d}\xi \, \frac{\Omega^2(\xi)(m(\xi)/M)}{s^2 + \Omega^2(\xi)} = s^2 \int_0^1 \mathrm{d}\xi \, \frac{\mu}{(1 - \xi)^2 + \xi^2} \frac{\Omega_0^2 \xi^2/(1 - \xi)^2}{s^2 + \Omega_0^2 \xi^2/(1 - \xi)^2} \tag{19}$$

$$= s^2 \mu \Omega_0^2 \int_0^1 \frac{\mathrm{d}\xi}{(1 - \xi)^2} \frac{\xi^2/(1 - \xi)^2}{(1 + \xi^2/(1 - \xi)^2)(s^2 + \Omega_0^2 \xi^2/(1 - \xi)^2)} \tag{20}$$

$$= s^2\mu\Omega_0^2 \int_0^\infty \mathrm{d}\theta \frac{\theta^2}{(1+\theta^2)(s^2+\Omega_0^2\theta^2)} \quad (21)$$

$$= s^2\mu\Omega_0^2 \int_0^\infty \mathrm{d}\theta \frac{1}{\Omega_0^2 - s^2}\left(\frac{1}{1+\theta^2} - \frac{s^2}{s^2+\Omega_0^2\theta^2}\right) \quad (22)$$

$$= [(s^2\mu\Omega_0^2/(\Omega_0^2 - s^2)](\pi/2)(1 - s/\Omega_0). \quad (23)$$

To obtain equation (23) the integral formula

$$I(s) = \int_0^\infty \frac{\mathrm{d}\theta}{s^2+\Omega_0^2\theta^2} = \begin{cases} +\pi/2\Omega_0 s, & \mathrm{Re}\,\{s\} > 0, \\ -\pi/2\Omega_0 s, & \mathrm{Re}\{s\} < 0 \end{cases} \quad (24)$$

has been used. Since Re $\{s\} > 0$ on the Bromwich path which is used to invert the Laplace transforms in equations (9) and (10), the result $I(s) = +\pi/(2\Omega_0 s)$ is used in equation (23).

Substitution of equation (23) into equations (9) and (10) now gives, after some manipulation,

$$\tilde{x}_M(s) = (1 + s/\Omega_0)(v_0/\Omega_0^2)/[s^3/\Omega_0^3 + (s^2/\Omega_0^2)(1 + \mu\pi/2) + s/\Omega_0 + 1], \quad (25)$$

$$\tilde{x}(\xi, s) = \frac{(\Omega^2(\xi)/\Omega_0^2)(1 + s/\Omega_0)v_0/\Omega_0^2}{(s^2/\Omega_0^2 + \Omega^2(\xi)/\Omega_0^2)(s^3/\Omega_0^3 + s^2/\Omega_0^2(1+\mu\pi/2) + s/\Omega_0 + 1)}. \quad (26)$$

It is easy to show using the Routh stability criterion [6] that the roots of the cubic polynomial in the denominator of equation (25) lie in the left half of the complex s-plane for all positive values of μ. The time domain response $x_M(t)$ therefore damps exponentially. The roots of the denominator of equation (26) are the left half-plane roots of the cubic, plus the two purely imaginary roots corresponding to the quadratic factor. The response $x(\xi, t)$ therefore consists of exponentially decaying terms plus an undamped sinusoidal term.

It is possible to give explicit formulas for the inverse Laplace transforms corresponding to equations (25) and (26), but the expressions are long. Instead, Figure 2 shows plots of $x_M(t)$ and $x(\xi, t)$ for $\mu\pi/2 = 0{\cdot}2$ and for three particular values of ξ. The displacement $x_M(t)$ decays exponentially to zero, with time constants determined by the real parts of the roots of the cubic equation in the denominator of equation (25). The distributed oscillator displacement $x(\xi, t)$ is identically zero at $\xi = 0$, since $k(\xi) = 0$ for $\xi = 0$, and thus there is no coupling to the primary oscillator. As $\xi \to 1$, $k(\xi)$ approaches infinity (rigid coupling to the primary oscillator) and thus $x(\xi, t) \to x_M(t)$ as $\xi \to 1$. For $0 < \xi < 1$, $x(\xi, t)$

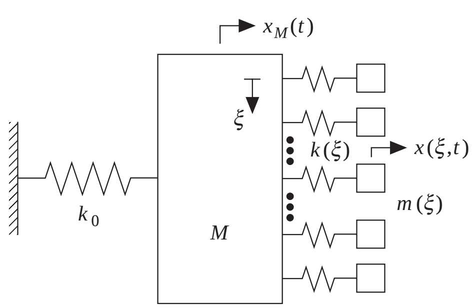

Figure 1. Primary mechanical oscillator connected to continuously distributed secondary oscillators.

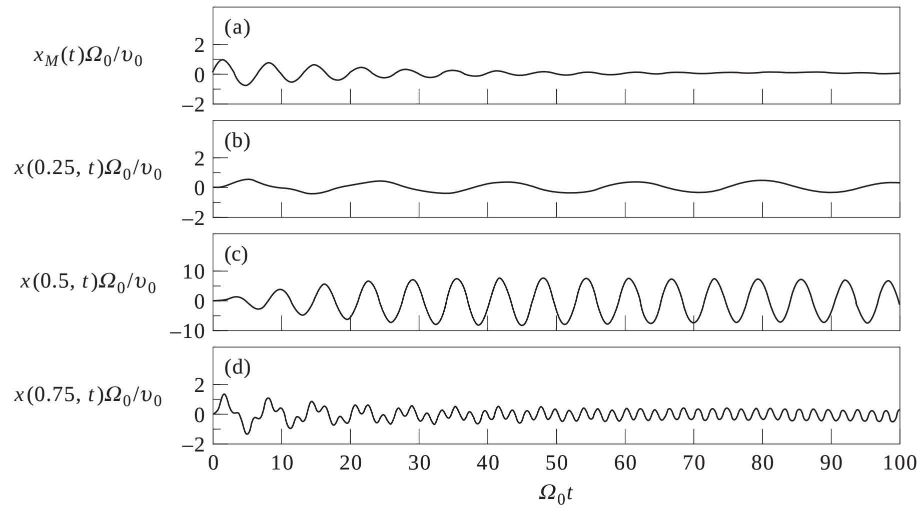

Figure 2. Dynamic responses (a) $x_M(t)$ and $x(\xi, t)$ for $\mu\pi/2 = 0{\cdot}2$ and (b) $\xi = 0{\cdot}25$, (c) $0{\cdot}5$, (d) $0{\cdot}75$.

attains a steady state oscillation, as discussed above, and as shown by the responses for the three values $\xi = 0{\cdot}25$, $\xi = 0{\cdot}5$ and $\xi = 0{\cdot}75$ shown in Figure 2.

The system described by equations (1) and (2) is conservative; equations (1) and (2) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}M\dot{x}_M^2(t) + \tfrac{1}{2}k_0 x_M^2(t) + \int_0^1 \mathrm{d}\xi\,[\tfrac{1}{2}m(\xi)\dot{x}^2(\xi, t) + \tfrac{1}{2}k(\xi)(x(\xi, t) - x_M)^2]\right\} = 0, \qquad (27)$$

so that the total energy

$$E = \tfrac{1}{2}M\dot{x}_M^2(t) + \tfrac{1}{2}k_0 x_M^2(t) + \int_0^1 \mathrm{d}\xi\,[\tfrac{1}{2}m(\xi)\dot{x}^2(\xi, t) + \tfrac{1}{2}k(\xi)(x(\xi, t) - x_M)^2] \qquad (28)$$

is constant. For the initial conditions assumed here, the total energy is

$$E = \tfrac{1}{2}Mv_0^2. \qquad (29)$$

As a check on the solution for $x_M(t)$ and $x(\xi, t)$, the energy E given by equation (28) is computed as $t \to \infty$. As $t \to \infty$, $x_M(t)$ and $\dot{x}_M(t)$ go to zero, and $x(\xi, t)$ can be obtained by evaluating the contribution to the inverse Laplace transform of the purely imaginary roots of the quadratic factor in the denominator of equation (26). This gives

$$x_{ss}(\xi, t) = (v_0/\Omega_0^2)A(\xi)\,\mathrm{e}^{\mathrm{i}\Omega(\xi)t} + \text{c.c.}, \qquad (30)$$

where

$$A(\xi) = \frac{1}{2\mathrm{i}}\,\frac{(\Omega(\xi)/\Omega_0)(1 + \mathrm{i}(\Omega(\xi)/\Omega_0))}{-\mathrm{i}(\Omega^3(\xi)/\Omega_0^3) - (\Omega^2(\xi)/\Omega_0^2)(1 + \mu\pi/2) + \mathrm{i}(\Omega(\xi)/\Omega_0) + 1}. \qquad (31)$$

The quantity $x_{ss}(\xi, t)$ is the steady state ($t \to \infty$) value of $x(\xi, t)$, and c.c. denotes the complex conjugate. The steady-state value of the total energy E is

$$E_{ss} = \int_0^1 \mathrm{d}\xi \, [\tfrac{1}{2} m(\xi) \dot{x}_{ss}^2(\xi, t) + \tfrac{1}{2} k(\xi) x_{ss}^2(\xi, t)] \tag{32}$$

$$= \int_0^1 \mathrm{d}\xi \, [\tfrac{1}{2} m(\xi) \dot{x}_{ss}^2(\xi, t) + \tfrac{1}{2} \Omega^2(\xi) m(\xi) x_{ss}^2(\xi, t)]. \tag{33}$$

Using equation (30), the steady state energy in equation (33) is

$$E_{ss} = \frac{v_0^2}{\Omega_0^2} \int_0^1 \mathrm{d}\xi \; 2m(\xi)\Omega^2(\xi)|A(\xi)|^2. \tag{34}$$

Substitution of the expressions for $m(\xi)$ and $A(\xi)$ given by equations (13) and (31) into equation (34) gives

$$E_{ss} = \frac{v_0^2}{\Omega_0^2} \int_0^1 \mathrm{d}\xi \, \frac{2\mu M}{(1-\xi)^2 + \xi^2} \frac{\Omega^2(\xi)}{4}$$

$$\times \left| \frac{(\Omega(\xi)/\Omega_0)(1 + \mathrm{i}(\Omega(\xi)/\Omega_0))}{-\mathrm{i}(\Omega^3(\xi)/\Omega_0^3) - (\Omega^2(\xi)/\Omega_0^2)(1 + \mu\pi/2) + \mathrm{i}(\Omega(\xi)/\Omega_0) + 1} \right|^2 \tag{35}$$

$$= \frac{v_0^2}{\Omega_0^2} \int_0^1 \frac{\mathrm{d}\xi}{(1-\xi)^2} \frac{\mu M}{1 + \xi^2/(1-\xi)^2} \frac{\Omega^2(\xi)}{2}$$

$$\times \left| \frac{(\Omega(\xi)/\Omega_0)(1 + \mathrm{i}(\Omega(\xi)/\Omega_0))}{-\mathrm{i}(\Omega^3(\xi)/\Omega_0^3) - (\Omega^2(\xi)/\Omega_0^2)(1 + \mu\pi/2) + \mathrm{i}(\Omega(\xi)/\Omega_0) + 1} \right|^2. \tag{36}$$

Again making the change of variables $\theta = \xi/(1-\xi)$, equation (36) becomes

$$E_{ss} = \frac{v_0^2}{\Omega_0^2} \int_0^\infty \mathrm{d}\theta \, \frac{\mu M}{1+\theta^2} \frac{\Omega_0^2\theta^2}{2} \left| \frac{(1 + \mathrm{i}\theta)\theta}{-\mathrm{i}\theta^3 - \theta^2(1 + \mu\pi/2) + \mathrm{i}\theta + 1} \right|^2 \tag{37}$$

$$= \tfrac{1}{2} M v_0^2 \mu \int_0^\infty \mathrm{d}\theta \left| \frac{\theta^2}{-\mathrm{i}\theta^3 - \theta^2(1 + \mu\pi/2) + \mathrm{i}\theta + 1} \right|^2. \tag{38}$$

The integral in equation (38) can be evaluated using the tabulated results in reference [7]; the result is

$$E_{ss} = \tfrac{1}{2} M v_0^2 \mu (1/\mu) = \tfrac{1}{2} M v_0^2, \tag{39}$$

which confirms that the initial kinetic energy of the mass M equals the energy in the steady state vibration of the distributed oscillators.

If the number of secondary oscillators in Figure 1 is finite, the system consisting of the primary plus secondary oscillators has a finite number of real natural frequencies, and the response to any set of initial conditions is a linear combination of undamped oscillations at the natural frequencies of the system. The response of such a system is quasi-periodic [8], and such a system cannot exhibit damping. As the distribution of secondary oscillators becomes continuous, the distribution of the natural frequencies of the system may also be thought of as becoming continuous, and the continuous distribution of undamped oscillations may be superposed to produce a response with exponential damping.

The set of continuously distributed oscillators considered here may also be interpreted as a "heat reservoir." The initial mechanical energy of the primary oscillator is transformed into steady state vibration of the set of secondary oscillators ("heat"), and never returns to the primary system as mechanical energy.

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